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Espresso

A sip of filtered research

CAMBRIDGE Mathematics

Tabitha Gould, Lucy Rycroft-Smith & Fran Watson, 2024

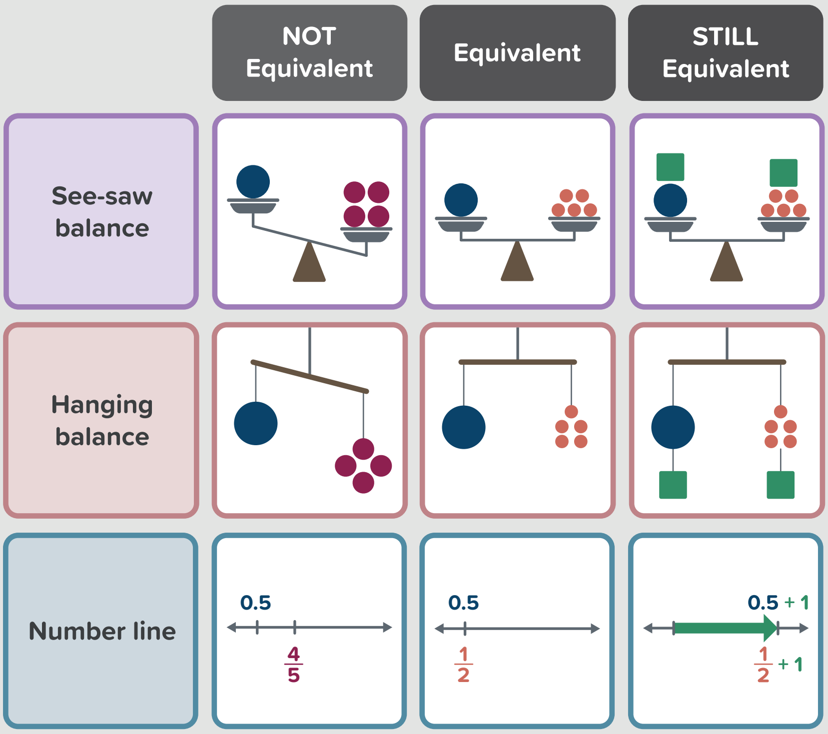
Talking point

What does research suggest about teaching and learning equivalence?

In summary

- Equivalence can be understood as a relationship between two things that can be swapped for one another for a specific purpose
- Equivalence is not just part of learning in number or algebra; it is useful across mathematics, although it is also context-dependent
- An understanding of equivalence involves using it to solve problems, to maintain balance or equality, to comprehend something differently, and using the equal sign as a symbol of a relationship
- Understanding equivalence can be supported by recognising that objects and processes in mathematics can be represented in interchangeable ways, and recognising when they are not equivalent
- Noticing equivalence allows students to substitute one form or representation with another, which helps to solve a problem, including simplifying
- Students can learn about equivalence from a very early age, and supporting students in understanding equivalence increases mathematics achievement
- It is suggested that students use the balance principle (either see-saw balance or hanging balance) and number line models to support exploration of equivalence, as well as searching for similarities between processes (as in the infographic)

Useful models of equivalence relationships



1 The concept of equivalence is, surprisingly, not well agreed-upon in mathematics education, especially alongside other ideas such as equality and similarity.[1] One useful definition of equivalence refers to two quantities, processes, objects, or measures that can be exchanged for one another in a particular context or for a specific purpose.[2] Equivalence is one of the most pervasive and foundational concepts in mathematics, one which emerges in many different mathematical areas.[1] However, there is a tension between the concept of equivalence as powerful and unifying,[3] and the way it is used in different contexts, in particular the different levels of 'ignorable difference' that are acceptable.[1] For example, whether or not two measures are considered equivalent depends upon the accuracy of the physical tool(s) and how the measurements are to be used;[4] in statistics, a dataset can be considered equivalent to a model if the differences are small enough to be ignored for modelling purposes.[5] More research on equivalence is needed, alongside better ways to support teachers to use that research in practice.[1]

Implications:

Since definitions of equivalence are not always clear, it may be useful to think of it as a relationship between two things that can be substituted for one another for a specific purpose

The mathematical idea of equivalence is not just part of learning in number or algebra; it is applicable and useful across mathematics

If teachers think about equivalence as both pervasive and context-dependent, it may help them to explore its usefulness with students

2 Some research[6] suggests children's understanding of mathematical equivalence includes problem solving (for example, 3 + 5 + 6 = 3 + __) and being able to explain meanings of the equal sign. Understanding mathematical equivalence involves not only understanding the equal sign as a relational symbol but also using symbol sense and relational thinking; for example, recognising the two sides of an equation (or any statement of equivalence) as mathematical objects (i.e., interpreting 4 + 3 not only as the operational process of adding 3 to 4 but also as an abstract object in its own right that can be manipulated), and then recognising that objects in mathematics, including numbers and expressions, can be represented in a variety of interchangeable ways.[7] An understanding of equivalence also includes recognising when things are *not equivalent*.[8]

Implications:

An understanding of equivalence involves using it to solve problems, and using the equal sign as a symbol of a relationship

Recognising that objects in mathematics (like numbers or expressions) can be represented in a variety of interchangeable ways supports the development of an understanding of equivalence

3 It is important that students understand why equivalence might be useful: for example, to substitute one form or representation for another which helps to solve a problem; to maintain balance or equality; or to comprehend something differently, including simplifying.[3] Additionally, using equivalent forms of numbers or expressions can help to unify these forms into one idea – such as the fraction, ratio and percentage form of a proportion – because "equivalence means that one can use any member of a class of equivalent numbers and, in fact, should select whichever representation is most applicable to the task at hand."[3(p130)] Recognising equivalent structures within things that are otherwise different also allows students to use analogical reasoning (this thing is like that thing) and to simplify, zoom in or out, or abstract.[9] Also important is recognising that different processes can be considered equivalent if they give the same outcome,[8] which supports ideas such as transformations, simulations, algebraic reasoning, modelling and conversions between measures.

Implications:

Understanding equivalence allows students to substitute one form or representation for another which helps to solve a problem, to maintain a relationship or equality, or to comprehend something differently, including simplifying

Equivalence can be used to solve problems by: understanding that different representations can stand for the same thing; recognising features or structures that are the same; recognising that different processes can lead to the same outcome

4 Learning about equivalence from a very early stage, even before the equal sign is introduced, can support understanding relationships between numbers, numerical and algebraic expressions,[10] and increase mathematics achievement.[7] Early informal experiences with equivalence should include exploring equivalent and non-equivalent situations, using the balance principle to model maintaining equivalence, either see-saw balances[8] or hanging balances,[11] as well as using number lines to investigate inverse and equivalent processes[8] (as in the infographic). Subsequently, encouraging students to search for structural similarities between processes is also suggested as useful; for example, "what is the same in the processes for '34 – 16' and '3 weeks 4 days subtract 1 week 6 days'."[8(p90)]

Implications:

Students can learn about equivalence from a very early age, and supporting students in understanding equivalence increases mathematics achievement

It is suggested that students use the balance principle (either see-saw balances or hanging balances) and number line models, as well as searching for structural similarities between processes, to support exploration of equivalence

> "Recognizing equivalence is a more significant obstacle for students than we and their teachers had expected"
>
> Ball et al., 2003[12(p15)]

> "Equivalence has had many different faces and [...] many different names"
>
> Asghari, 2019[13(p4675)]

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