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Espresso

A sip of filtered research

CAMBRIDGE Mathematics

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Talking point

What does research suggest about the early development of functional thinking?

In summary

- Functional thinking is one important strand of early encounters with algebraic ideas, emphasising, justifying and reasoning with relationships and working with different representations of them
- Functional thinking involves exploring recursive patterns, covariation and correspondence by considering relationships between two data sets
- Children as young as 5 can enjoy opportunities for functional thinking
- When developing functional thinking, it is useful for students to work with generalising growing patterns using number cards to show position, and also using context (such as children's literature), multiple representations and technological tools
- Functional thinking in mathematics may not be well understood by teachers, so good-quality professional learning is needed, helping them to explore tasks as learners and then as teachers, and using diagrams and images wherever possible
- It is helpful in developing functional thinking to help students express relationships in words, tables (especially t-charts), diagrams, function machines, graphs and symbols (both conventional and invented) and reasoning with them, as well as moving between them fluently

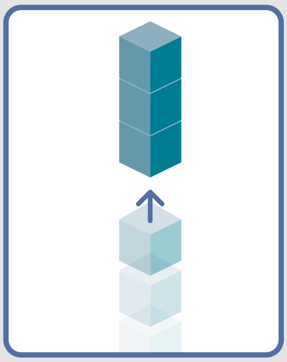
Transforming questions into functional thinking tasks

1 Instead of just

I have built a tower of 3 multilink cubes. What is its surface area (don't forget to include the bottom)?

2 You could add

What would the surface area be for a tower like this that was 50 multilink cubes tall? As I build taller and taller towers, how does the surface area/number of square faces on the 'outside' change?



3 Which might prompt

I think it gets five more added each time, but I'm not sure about the middle part
RECURSIVITY

The number of outside squares is the number of cubes times four (the middle ones) plus two (the top and bottom)
CORRESPONDENCE

Every time you add a new cube to the tower it adds five more surface area squares and takes away one
COVARIATION

I think I have found a rule – you take the number of cubes and multiply by three ... but then you have to take away the hidden ones in the middle
EXPLORING STRUCTURE

I have found how many for 1, 2 and 3 cubes ... I'm going to draw a picture
COMPUTING & RECORDING

I'm going to make a table and look for patterns. I think the number of faces goes up by the same amount each time
ORGANISING & CONJECTURING

Adapted from ideas in Blanton & Kaput, 2011[1]

1 Teaching and learning what is often called "algebra" in school mathematics (although this is not always agreed on as a term)[2] is difficult. Some research suggests a useful approach to early algebra is to integrate three important strands: generalised arithmetic; equivalence, expressions, equations and inequalities; and functional thinking.[3] Functional thinking focuses on the relationship between two or more varying quantities and has been suggested as a way to improve understanding in learning algebra for students[4] by helping students understand and use the unifying concept of functions to see and use algebraic structure more flexibly and less procedurally.[5] Functional thinking has been defined as exploring recursive patterning, covariation and correspondence[1] (see infographic) and also as exploring relationships both within and between sets of data[6] as well as justifying and reasoning with those relationships.[7] Other definitions focus on the importance of representation in functional thinking – creating or using systems of representation and practising allowable actions on them – as opposed to only manipulating symbols.[8] Children can be supported in successfully working with functional thinking from the early years of schooling (age 5).[1]

Implications:

Functional thinking is an important strand of teaching and learning early algebra, emphasising relationships and different representations of them, and justifying and reasoning with those relationships

Functional thinking involves exploring recursive patterns, covariation and correspondence by considering relationships within and between two data sets, especially thinking about ways to represent these relationships

Children as young as 5 can meaningfully engage with opportunities for functional thinking

2 It is suggested that the interactions of context, multiple representational forms and technological tools are helpful to support students' development of functional thinking,[9] as well as using a learning environment based on experiments with real materials (manipulatives) or simulations.[10] In one study,[4] researchers give an example of using meaningful contexts such as literature to explore functional thinking: they designed lessons around the Chinese folk tale *Two of Everything* which features a large brass pot that doubles everything placed in it.[11] One possibility to support students in exploring functional thinking is generalising of growing patterns and number sequences[3] because it provides a powerful vehicle for understanding the dependent relations between quantities that are the basis for mathematical functions.[12] It is important that growing patterns are expressed in relation to their position in the pattern; this can be done by using position number cards which helps pupils view the number of the card as a sign pointing to the position of each image.[13]

Implications:

Functional thinking can be learned and taught from a very young age; for example, using children's literature to provide context

It is useful for students to work with generalising growing patterns and number sequences when developing functional thinking, as well as using context, multiple representational forms and technological tools

When working with growing patterns, it is suggested that position number cards or similar are provided to help students notice the position of each instance of the pattern

3 There is evidence that some teachers do not yet have a good understanding of functional thinking, especially at the primary level[14] or at the beginning of their career,[15] and so good-quality support and professional learning is needed. It is important that teachers are able to explore functional thinking tasks first as learners and then as teachers.[3] In one study of student teachers, those that were most successful at functional thinking tasks used drawings and images to demonstrate and explain the revealed relationships.[15]

Implications:

Functional thinking in mathematics may not be well understood by teachers and so good-quality professional learning for teachers is needed

It may help teachers to explore functional thinking tasks first as learners and then as teachers, and also to use drawings or images wherever possible

4 Central to a functional thinking perspective is supporting students in expressing the relationships between covarying quantities and reasoning with multiple representations such as words, tables, diagrams, graphs, or symbols using conventional or unusual/created symbols.[16] The ability to create, interpret, translate between and connect multiple representations (representational fluency) is also important.[1] Useful tools to support functional thinking include the t-chart or t-table, a simple table with two variables in ordered corresponding columns where students can represent and explore instances of two varying quantities together,[4] and the function machine – an input-output box metaphor that helps students conjecture and explore functions as processes.[5]

Implications:

It is helpful in developing functional thinking to help students express relationships in words, tables, diagrams, function machines, graphs and symbols (both conventional and invented) and reason with them

One suggested way of working with functional thinking is the t-chart, a simple two-column table to record and reason with two variables; another is the function machine, an input-output box analogy

> "Functional thinking doesn't happen overnight. It starts with the introduction of a good task and builds with the organization of data so students can see clearly where patterns and relationships lie"
>
> **Murphy Gardiner & Sawrey, 2016[17]**

> "Not only are young children capable of thinking functionally, but also of representing this thinking in ways that we never dreamed primary aged children could"
>
> **Warren & Cooper, 2006[6(pp13–14)]**

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