Espresso
CoffeePod

CAMBRIDGE Mathematics

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Transcript for CoffeePod, based on Espresso 43

Welcome!

This is an audio recording of Espresso 43, which was published in August 2022, and written by Fran Watson, Lucy Rycroft-Smith and Tabitha Gould.

Espressos are 2-page filtered research documents from Cambridge Mathematics. Each one focuses on selecting and summarising research on a given area of mathematics education, called a Talking Point. They are published as free pdf documents on the Cambridge Mathematics website www.cambridgemaths.org.

This CoffeePod is 23 minutes and 41 seconds long in total. It comprises the following sections: a talking point; quotes; a summary; the main text of the Espresso; the infographic; and an optional reference section which is also available as a separate document linked in the video notes.

You will hear the following sound to indicate a reference [tone], followed by the reference number to help you identify it.

The chapter timings for the different sections, and a link to the original published Espresso, can also be found in the video notes.

You can give us feedback or ask questions in the comments section under this video – we'd love to hear from you!

Talking point:

What does research suggest about the teaching around factors, multiples and prime numbers?

Here are two quotes from the research that you might find interesting to think about:

"That's part of what makes primes so interesting: not only is the number line studded with primes all the way up to infinity, but that the whole number line can be produced using nothing but primes."

This was written in 2020 by Graham Templeton in an article on Mashable.com called 'Why should we care about prime numbers?'[1]

"The operation of addition or subtraction is the same as extending or cutting off lines; the product of two numbers a and b is the same as the geometric construction of a rectangle having adjacent sides a and b."

This was written in 2014 by Joseph Mazur in a book called 'Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers'; the quotation can be found on page 143 of the book.[2]

In summary:

We found seven main implications for teachers in the research. We have numbered them here for ease of listening but they are not intended to be hierarchical (that is, no one implication is more important than any of the others). The seven implications are:

1. Decomposing numbers to investigate their multiplicative structure can support a flexible approach to problem solving and should come before the introduction of rules or procedures
2. Activities in which students sort objects into regular arrays of width two and explore which numbers can be split into two equal groups, and also into equal groups of two, all support conceptual understanding that even numbers are divisible by two
3. Students need to link doubling to multiplying in order to appreciate when it will always result in an even number
4. Visualising building numbers by scaling or growing, rather than by repeated addition, helps support multiplicative reasoning and generalising
5. Working with characteristics of primes can help avoid misconceptions about their size and prevalence as factors of other numbers
6. Practising seeing prime factors both individually and in combinations, can help support flexible reasoning about the divisibility of the entire number

7. Making links between different methods of finding the lowest common multiple of two numbers can support conceptual understanding; Venn diagrams are suggested as useful ways to visualise common prime factors of two numbers

Here is more detail about the research which fed into the summary implications, organised into five paragraphs which each have a set of implications for teachers at the end.

Paragraph one:

Developing an awareness of the underlying multiplicative structure of whole numbers can support a flexible approach to problem solving.[3] Creating and considering equivalent representations of the same number (which is discussed in the infographic section) and the related concepts of factors, multiples and divisors, can reveal general properties about families of numbers such as odd, even, composite and prime.[4] Research is scarce on school students' experience of using primes to build numbers, but some exists on teachers' learning of and teaching around these concepts.[5,6,7]

There are two implications of paragraph one:

First implication: Decomposing whole numbers to investigate their structure can support a flexible approach to problem solving

Second implication: Considering the multiplicative structure of numbers can highlight generalisations about their properties

Paragraph two:

Odd and even numbers are often some of children's first experiences of seeing the multiplicative structure of numbers. It is suggested that activities in which students physically sort groups of objects into regular arrays of width two can support a recognition of the way even numbers can form a rectangle and odd numbers can form an L-shape, and this supports pattern-spotting and generalisation.[8] Providing the opportunity to explore which numbers are made of pairs, can be split into two equal groups, and can be split into equal groups of two is more helpful than a reliance on identifying the last digit of a number as a characterisation of the 'two-times-table'.[9] Students do not necessarily connect 'doubling' with multiplying by two,[10] or recognise that when operating on whole numbers, doubling will always produce an even number, and so need support to do so.[11]

There are four implications of paragraph two:

First implication: Activities in which students sort objects into twos can helpfully lead to the visualisation and then generalisation that even numbers can form a rectangle and odd numbers an L-shape

Second implication: Exploring which numbers can be split into two equal groups and also into equal groups of two supports conceptual understanding that even numbers are divisible by two

Third implication: Learning a list of digit endings as a way of identifying odd and even numbers does not support conceptual understanding of whether or not the number has a factor of two

Fourth implication: Doubling needs to be explicitly linked to multiplying for students to appreciate when it will always result in an even number

Paragraph three:

A variety of representations and contexts are suggested for exploring multiples (numbers that divide exactly by another without a remainder), with the goal being for students to identify patterns and be able to generalise characteristics.[12] Conceptual understanding of divisibility should be established before introducing divisibility rules and procedures.[6] Vocabulary needs careful attention, as reasoning with and about multiplicative number sentences can lead to words such as ‘multiple’ (an object) and ‘multiply’ (a process) being mistakenly used interchangeably.[5] Development of multiplicative reasoning can be blocked if multiples are visualised via repeated addition rather than scaling or growing and this creates difficulties in coming to understand a number’s structure as a product of its prime factors.[13]

There are four implications of paragraph three:

First implication: Using a range of representations and contexts when exploring multiples supports students’ pattern-spotting and generalising

Second implication: Conceptual understanding of divisibility should be established before rules or procedures are introduced

Third implication: Careful consideration of the specific vocabulary being used (such as odd, even, factor and multiple) is recommended

Fourth implication: Visualising composing numbers multiplicatively (by scaling or growing) rather than additively helps support multiplicative reasoning and generalising

Paragraph four:

A tendency to define primes by what they are not[7] can result in misconceptions that prime numbers are small (under 100) and that every large number, if composite (that is, having more than two factors), is divisible by a small prime number. Students have a tendency to calculate the product of prime factors and then divide the answer to check for divisibility, rather than reasoning flexibly.[6]

There are two implications of paragraph four:

First implication: Working with characteristics of primes (rather than what they are not) can help avoid misconceptions about their size and prevalence as factors of other numbers

Second implication: Practising seeing prime factors both individually and in combinations can help support flexible reasoning about the divisibility of the entire number

Paragraph five:

There are three common approaches to finding the lowest common multiple:

a. set intersection – finding a number that appears in a list of multiples for each;

b. creating a multiple and then dividing – checking for divisibility by the second number through an ordered list of multiples of the first; and

c. prime factorisation – finding the minimal product of prime powers that contain both their factorisations.

However, students often seem unable to recognise the equivalence of these methods.[13] A useful suggested approach to finding the highest common factor of two numbers is using a Venn diagram representation, where each circle (or set) contains the prime factors of its respective number, with any shared prime factors placed in the overlap. These shared prime factors can then be multiplied to find the highest common factor.[4]

There are two implications of paragraph five:

First implication: Making links between different methods of finding the lowest common multiple of two numbers can support conceptual understanding relating to factors and multiples, and also flexible problem-solving

Second implication: A Venn diagram can be used to visualise common prime factors of two numbers, which can then be multiplied together to find the highest common factor

The Espresso also has an infographic, titled 'Recognising 24'.

There are many ways to 'see' a number, and it is useful for both teachers and students to visualise or think about numbers in terms of their properties. This infographic incorporates different ways of seeing a number, in this case 24.

When you think about the number 24, what springs to mind? Can you imagine it as an array (a neat set of groups)? How many groups are there, and how big is each group? Can you imagine it first as two, doubled, then doubled again, and then multiplied by three? What could this look like? For us, it could be a kind of three-armed windmill. You might like to think about how you could fold and un-fold a piece of paper to reveal each of your steps as a multiplication or division.

What about imagining 24 as a flat square windowpane that grows in all directions, first by a factor of two, then two again, and then three? Can you imagine each stage of this growing as a stack of flat windowpanes, each one larger than the last?

How about imagining 24 as a tree or plant, and breaking it down into its factors as each root branches downwards into the soil? The numbers at the ends of the first branching are two multiplied by twelve, then further breaking down the twelve root into two multiplied by six, then breaking down the six root into two multiplied by three. So that at the four final ends of the roots are the factors two, two, two and three.

What about imagining a series of towers stacked up that all have a height of 24? What might the stacks look like? Can you see a tower with 24 blocks, all the same size? Is there another tower of the same height but this time made from only twelve identical blocks? What other block sizes might make a tower of equal height to these two?

Thanks for sampling this new product.

We loved you joining us.

The details of the references are coming next, so feel free to listen or stop here – whichever you prefer. If you enjoyed this CoffeePod, there are others available, and more recordings in progress.

We'd love to hear your thoughts about this CoffeePod. Please email us at admin@cambridgemaths.org. And you can also read the original Espresso by going to www.cambridgemaths.org.

The Espresso has a reference list of 13 entries written using APA 7 notation.

The references can either be accessed as a separate document or in the published Espresso via links in the video information box. Alternatively, you can continue to listen to them here.

Reference 1: Templeton, G. (2020, April 29). *Why should we care about prime numbers?* Mashable.com.

Reference 2: Mazur, J. (2014). *Enlightening symbols: A short history of mathematical notation and its hidden powers*. Princeton University Press.

Reference 3: Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10–32.

Reference 4: Kurz, T. L., & Garcia, J. (2012). The complexities of teaching prime decomposition and multiplicative structure with tools to preservice elementary teachers. *Journal of Research in Education, 22*(2), 169–192.

Reference 5: Zazkis, R. (2000). Factors, divisors, and multiples: Exploring the web of students' connections. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. 4, pp. 210–238). American Mathematical Society.

Reference 6: Zazkis, R., & Campbell, S. (1996). Prime decomposition: Understanding uniqueness. *The Journal of Mathematical Behavior, 15*(2), 207–218.

Reference 7: Zazkis, R., & Liljedahl, P. (2004). Understanding primes: The role of representation. *Journal for Research in Mathematics Education, 35*(3), 164–186.

Reference 8: Griffiths, R., Gifford, S., & Back, J. (2016). *Making numbers: Using manipulatives to teach arithmetic*. Oxford University Press.

Reference 9: Zazkis, R. (1998). Odds and ends of odds and evens: An inquiry into students' understanding of even and odd numbers. *Educational Studies in Mathematics, 36*(1), 73–89.

Reference 10: Harel, G., &. Confrey J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. SUNY Press.

Reference 11: Ryan, J., & Williams, J. (2007). *Children's mathematics 4–15: Learning from errors and misconceptions*. McGraw-Hill Education (UK).

Reference 12: van den Heuvel-Panhuizen, M. (Ed.). (2008). *Children learn mathematics: A learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school*. Sense Publishers.

Reference 13: Brown, A., Thomas, K., & Tolias, G. (2002). Conceptions of divisibility: Success and understanding. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (Vol. 2, pp. 41–82). Ablex Publishing.